

MAXIMUM & MINIMUM VALUES OF TRIGONOMETRICAL EXPRESSIONS

Section - 4

In this section, we shall discuss problems on finding the maximum and minimum values of various trigonometrical expressions.

As discussed that $-1 \leq \sin x \leq 1$, $-1 \leq \cos x \leq 1$, $-\infty < \tan x < \infty$, $|\sec x| \geq 1$ and $|\operatorname{cosec} x| \geq 1$.

Consider the expression $a \cos \theta \pm b \sin \theta$, where θ is a variable.

Let $y = a \cos \theta \pm b \sin \theta$

Further, let $a = r \cos \alpha$ and $b = r \sin \alpha$. Then, $r = \sqrt{a^2 + b^2}$ and $\tan \alpha = \frac{b}{a}$

$$\therefore y = r \cos \alpha \cos \theta \pm r \sin \alpha \sin \theta \Rightarrow y = r \cos(\theta \mp \alpha)$$

We know that $-1 \leq \cos(\theta \mp \alpha) \leq 1$ for all θ

$$\Rightarrow -r \leq r \cos(\theta \mp \alpha) \leq r \quad \text{for all } \theta$$

$$\Rightarrow -\sqrt{a^2 + b^2} \leq y \leq \sqrt{a^2 + b^2} \quad \text{for all } \theta$$

$$\Rightarrow -\sqrt{a^2 + b^2} \leq a \cos \theta \pm b \sin \theta \leq \sqrt{a^2 + b^2} \quad \text{for all } \theta$$

It follows from the above discussion that $-\sqrt{a^2 + b^2}$ and $\sqrt{a^2 + b^2}$ are minimum and maximum values of $a \cos \theta \pm b \sin \theta$ for varying values of θ .

Note : Above result can also be derived by taking $a = r \sin \alpha$ and $b = r \cos \alpha$.

Important : The maximum and minimum values of $a \cos \theta \pm b \sin \theta + c$ are

$$c + \sqrt{a^2 + b^2} \quad \text{and} \quad c - \sqrt{a^2 + b^2}, \text{ respectively.}$$

$$\text{i.e.,} \quad c - \sqrt{a^2 + b^2} \leq a \cos \theta \pm b \sin \theta + c \leq c + \sqrt{a^2 + b^2}.$$

Illustrating the Concepts :

Find the maximum and minimum value of :

$$(i) \quad \sin \theta + \cos \theta \quad (ii) \quad \sqrt{3} \sin \theta - \cos \theta \quad (iii) \quad 5 \sin \theta + 12 \cos \theta + 7$$

Given expressions are in the form of :

$$a \sin \theta + b \cos \theta.$$

Express this in terms of one T-ratio by dividing and multiplying by $(a^2 + b^2)^{1/2}$

$$(i) \quad \sin \theta + \cos \theta = 1 \cdot \sin \theta + 1 \cdot \cos \theta$$

$$= \sqrt{2} \left(\frac{1}{\sqrt{2}} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta \right)$$

$$= \sqrt{2} \left(\sin \theta \cos \frac{\pi}{4} + \cos \theta \sin \frac{\pi}{4} \right)$$

$$= \sqrt{2} \sin \left(\theta + \frac{\pi}{4} \right)$$

Now $\sin$ of an angle must be between -1 and 1 .

$$\Rightarrow -1 \leq \sin \left(\theta + \frac{\pi}{4} \right) \leq 1$$

$$\Rightarrow -\sqrt{2} \leq \sqrt{2} \sin \left(\theta + \frac{\pi}{4} \right) \leq \sqrt{2}$$

So maximum value of $\sin \theta + \cos \theta$ is $\sqrt{2}$ and minimum value of $\sin \theta + \cos \theta$ is $-\sqrt{2}$.

$$(ii) \quad \sqrt{3} \sin \theta - \cos \theta = 2 \left(\frac{\sqrt{3}}{2} \sin \theta - \frac{1}{2} \cos \theta \right)$$

[Multiplying and divide by $\sqrt{(\sqrt{3})^2 + (1)^2} = 2$]

$$= 2 \left(\sin \theta \cos \frac{\pi}{6} - \cos \theta \sin \frac{\pi}{6} \right)$$

$$= 2 \sin \left(\theta - \frac{\pi}{6} \right)$$

$$\text{as } -1 \leq \sin \left(\theta - \frac{\pi}{6} \right) \leq 1 \Rightarrow -2 \leq 2 \sin \left(\theta - \frac{\pi}{6} \right) \leq 2$$

So maximum value is 2 and minimum value is -2.

(iii) Consider $5 \sin \theta + 12 \cos \theta$

$$= 13 \left[\frac{5}{13} \sin \theta + \frac{12}{13} \cos \theta \right]$$

[Multiplying and divide by $\sqrt{5^2 + 12^2}$]

Construct a triangle with sides, 5, 12, 13. If α is an angle of triangle,

Note : Above questions can also be solved using formula given in section-4.

then $\cos \alpha = 5/13$, $\sin \alpha = 12/13$,

$$5 \sin \theta + 12 \cos \theta = 13 [\sin \theta \cos \alpha + \cos \theta \sin \alpha]$$

$$5 \sin \theta + 12 \cos \theta + 7 = 13 [\sin (\theta + \alpha)] + 7$$

$$\text{as } -1 \leq \sin (\theta + \alpha) \leq 1$$

$$\Rightarrow -13 \leq 13 \sin (\theta + \alpha) \leq 13$$

$$-13 + 7 \leq 13 \sin (\theta + \alpha) + 7 \leq 13 + 7$$

So maximum value is 20 and minimum value is -6.

Illustration - 20 The expression $5 \cos x + 3 \cos (x + \pi/3) + 3$ lies between.

- (A) 4 and 10 (B) -4 and 10 (C) (0, 4) and (5, 10) (D) None of these

SOLUTION : (B)

$$5 \cos x + 3 \left(\cos x \cos \frac{\pi}{3} - \sin x \sin \frac{\pi}{3} \right) + 3$$

$$= \cos x \left(5 + \frac{3}{2} \right) - \sin x \cdot \frac{3\sqrt{3}}{2} + 3$$

$$= \frac{13}{2} \cos x - \frac{3\sqrt{3}}{2} \sin x + 3$$

$$= \sqrt{\frac{169}{4} + \frac{27}{4}} \left\{ \frac{13/2}{\sqrt{\frac{169}{4} + \frac{27}{4}}} \cos x - \frac{3\sqrt{3}/2}{\sqrt{\frac{169}{4} + \frac{27}{4}}} \sin x \right\} + 3$$

$$= 7 (\cos \alpha \cos x - \sin \alpha \sin x) + 3$$

$$= 7 \cos (\alpha + x) + 3$$

As $-1 \leq \cos (\alpha + x) \leq 1$

i.e., $-7 + 3 \leq 7 \cos (\alpha + x) + 3 \leq 7 + 3$

i.e., $-4 \leq 7 \cos (\alpha + x) + 3 \leq 10$

[Multiplying and divide by $\sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}$]

[where $\tan \alpha = \frac{3\sqrt{3}}{13}$]

Illustration - 21 If $\theta \in R$, the expression $a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta$ lies between :

(A) $\frac{-1}{2} \sqrt{b^2 + (a-c)^2}$ and $\frac{1}{2} \sqrt{b^2 + (a-c)^2}$

(B) $\frac{a+c}{2}$ and $\frac{a-c}{2}$

(C) $\frac{a+c}{2} - \frac{1}{2} \sqrt{b^2 + (a-c)^2}$ and $\frac{a+c}{2} + \frac{1}{2} \sqrt{b^2 + (a-c)^2}$

(D) None of these

SOLUTION : (C)

$$\text{Let } f(\theta) = a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta$$

$$= \frac{a(1 - \cos 2\theta)}{2} + \frac{b}{2} \sin 2\theta + \frac{c(1 + \cos 2\theta)}{2} = \left(\frac{c}{2} - \frac{a}{2}\right) \cos 2\theta + \frac{b}{2} \sin 2\theta + \frac{a}{2} + \frac{c}{2}$$

$$= \frac{1}{2}(a+c) + \left(\sqrt{b^2 + (a-c)^2}\right) (\sin 2\theta \cos \alpha - \cos 2\theta \sin \alpha) = \frac{1}{2}(a+c) + \frac{\sqrt{b^2 + (a-c)^2}}{2} \sin(2\theta - \alpha)$$

$$\text{As } -1 \leq \sin(2\theta - \alpha) \leq 1$$

$$\therefore \left\{\frac{a+c}{2}\right\} - \frac{\sqrt{b^2 + (a-c)^2}}{2} \leq f(\theta) \leq \left\{\frac{a+c}{2}\right\} + \frac{\sqrt{b^2 + (a-c)^2}}{2}$$

Note : Above question can also be solved by directly applying result given in Section-4.

Illustration - 22 Find the maximum and minimum values of $\sin^6 x + \cos^6 x$.

- (A) 1 and $\frac{1}{4}$ (B) 1 and $\frac{3}{4}$ (C) 0 and $\frac{1}{4}$ (D) 0 and $\frac{3}{4}$

SOLUTION : (A)

If the function contains only even powers of sine and cosine of the same angle, use the following properties

$$(i) \quad \sin^2 x + \cos^2 x = 1 \quad (ii) \quad 2 \sin x \cos x = \sin 2x.$$

$$\begin{aligned} \text{Let } f(x) &= \sin^6 x + \cos^6 x \\ &= (\sin^2 x)^3 + (\cos^2 x)^3 \\ &= (\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x (\sin^2 x + \cos^2 x) \\ &= 1 - \sin^2 x \cos^2 x \\ &= 1 - \frac{3}{4} (\sin 2x)^2 \end{aligned}$$

When $(\sin 2x)^2$ is minimum i.e. 0, then y will be maximum and when $(\sin 2x)^2$ is maximum i.e. 1, then y will be minimum.

$$\text{Hence } f(x)_{\max} = 1 - \frac{3}{4} \times 0 = 1 \quad \text{and} \quad f(x)_{\min} = 1 - \frac{3}{4} \times 1 = \frac{1}{4}.$$